



ELSEVIER

Available online at [www.sciencedirect.com](http://www.sciencedirect.com)

SCIENCE @ DIRECT®

Journal of Geometry and Physics 54 (2005) 427–453

JOURNAL OF  
GEOMETRY AND  
PHYSICS

[www.elsevier.com/locate/jgp](http://www.elsevier.com/locate/jgp)

# Deformations of semisimple bihamiltonian structures of hydrodynamic type

Si-Qi Liu, Youjin Zhang\*

*Department of Mathematical Sciences, Tsinghua University, Beijing 100084, PR China*

Received 14 September 2004; accepted 5 November 2004

Available online 15 December 2004

---

## Abstract

We classify in this paper infinitesimal quasitrivial deformations of semisimple bihamiltonian structures of hydrodynamic type.

© 2004 Elsevier B.V. All rights reserved.

PACS: 02.40.k; 02.90.+p

MSC: 37K10; 35Q53

JGP SC: Classical integrable systems; Integrable PDEs

Keywords: Bihamiltonian structure; Infinitesimal deformation; Poisson cohomology

---

## 1. Introduction

A bihamiltonian structure of hydrodynamic type defined on the formal loop space of a manifold  $M$  consists of two compatible Poisson brackets of the form

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x - y) + \Gamma_k^{ij}(u(x))u_x^k\delta(x - y), \quad i, j = 1, \dots, n. \quad (1)$$

Here  $n = \dim M$  and  $\det(g^{ij}(u)) \neq 0$ . Such type of Poisson brackets were introduced and classified by Dubrovin and Novikov during the 1980s of the last century [10–12], they were used to describe the hamiltonian structures of systems of hydrodynamic type. According

---

\* Corresponding author. Tel.: +86 1062771380; fax: +86 1062781785.

E-mail addresses: [lsq99@mails.tsinghua.edu.cn](mailto:lsq99@mails.tsinghua.edu.cn) (S.Q. Liu), [y Zhang@math.tsinghua.edu.cn](mailto:y Zhang@math.tsinghua.edu.cn) (Y. Zhang).

to the theory of Dubrovin and Novikov, the inverse of  $(g^{ij})$  must be a flat metric (here the ‘metric’ is not required to be positive definite) of the manifold  $M$ , and the coefficients  $\Gamma_k^{ij}$  be given by the contravariant components of the Levi–Civita connection of this metric. Two such Poisson brackets corresponding to two flat metrics  $(g_1^{ij})^{-1}, (g_2^{ij})^{-1}$  are compatible if these two metrics form a flat pencil [9]. The most well-known examples of bihamiltonian structures of hydrodynamic type are possessed by the Whitham equations (in particular, the dispersionless limit) of integrable evolutionary PDEs of KdV type [10–12,26].

In the present paper we study the problem of classification of deformations of a given bihamiltonian structure of hydrodynamic type, these deformations depend on a parameter  $\epsilon$  which is called the dispersion parameter. The deformed bihamiltonian structure has the form

$$\begin{aligned} \{u^i(x), u^j(y)\}_a &= g_a^{ij}(u(x))\delta'(x - y) + \Gamma_{k;a}^{ij}(u(x))u_x^k\delta(x - y) \\ &+ \sum_{m \geq 1} \sum_{l=0}^{m+1} \epsilon^m A_{m,l;a}^{ij}(u, u_x, \dots, u^{(m+1-l)})\delta^{(l)}(x - y), \quad a = 1, 2. \end{aligned} \tag{2}$$

Here  $A_{m,l;a}^{ij}$  are homogeneous differential polynomials of degree  $m + 1 - l$ , i.e. they depend polynomially on the jet coordinates  $u^{i,m} = \partial_x^m u^i$ ,  $m \geq 1$  with the assignment of degree  $\deg u^{i,m} = m$ , and the coefficients of these polynomials are smooth functions of  $u^1, \dots, u^n$ . The class of bihamiltonian structures of the form (2) that satisfy some additional conditions is classified in [14]. These additional conditions include the so called tau-symmetry property and the property of linearization of the Virasoro symmetries of the corresponding hierarchy of bihamiltonian evolutionary PDEs, they ensure the existence of tau functions for solutions of the hierarchy and the possibility of representing the Virasoro symmetries of the hierarchy by the action of an infinite number of linear differential operators on the tau functions. Under the assumption of semisimplicity that will be explained below, the moduli space of this class of bihamiltonian structures corresponds to that of the semisimple Frobenius manifolds [14]. Here we will study the class of deformed bihamiltonian structures of the form (2) without the restriction of these additional properties.

The bihamiltonian structures of hydrodynamic type under our considerations are assumed to be semisimple, i.e. the matrix  $(g_2^{ij}(u) - \lambda g_1^{ij}(u))$  is nondegenerate for any  $\lambda \in \mathbb{R}$  near any generic point of  $M$ , and the eigenvalues of the matrix  $(g_1^{ij})^{-1} g_2^{ij}$  are pairwise distinct, here  $(g_1^{ij})^{-1}, (g_2^{ij})^{-1}$  are the flat metrics corresponding to the given bihamiltonian structure. The simplest example of semisimple bihamiltonian structures of hydrodynamic type has the form

$$\{u(x), u(y)\}_1 = \delta'(x - y), \quad \{u(x), u(y)\}_2 = u(x)\delta'(x - y) + \frac{1}{2}u(x)\delta(x - y), \tag{3}$$

it is the dispersionless limit of the bihamiltonian structure of the KdV hierarchy [21,27,29]. In [25] Lorenzoni studied its deformations at the approximation up to  $\epsilon^4$ . He showed that the equivalence classes of all such deformations are parameterized by a smooth function  $s(u)$ ,

the bihamiltonian structure of the KdV hierarchy corresponds to the special deformation with a nonzero constant  $s(u)$ . Here the equivalence relation between deformations of a bihamiltonian structures of hydrodynamic type is defined in [14], two deformations of the form (2) are defined to be equivalent if they are related by a Miura transformation

$$u^i \mapsto u^i + \sum_{k \geq 1} \epsilon^k F_k^i(u; u_x, \dots, u^{(k)}), \quad i = 1, \dots, n, \quad (4)$$

where  $F_k^i$  are homogeneous *differential polynomials* of degree  $k$ , note that they are not required to depend polynomially on  $u^1, \dots, u^n$ . In particular, a deformation (2) is defined to be trivial if it is equivalent to the undeformed bihamiltonian structure. For the above example, when the function  $s(u)$  does not vanish, the corresponding deformation of the bihamiltonian structure (3) is nontrivial. Nevertheless, Lorenzoni proved that at the approximation up to  $\epsilon^4$  all such deformations are *quasitrivial*. The notion of quasitriviality was also introduced in [14], a deformed bihamiltonian structure of the form (2) is called quasitrivial if it can be obtained from the undeformed one by a transformation of the form

$$u^i \mapsto u^i + \sum_{k \geq 1} \epsilon^k G_k^i(u; u_x, \dots, u^{(m_k)}), \quad i = 1, \dots, n. \quad (5)$$

Here each function  $G_k^i$  depends rationally on the jet coordinates  $u_x, \dots, u^{(m_k)}$  for certain positive integer  $m_k$  and is homogeneous of degree  $k$ . In [14] it was proved that all semisimple bihamiltonian structures of the form (2) that satisfy the tau-symmetry property are quasitrivial. The method given there can in fact be employed to prove the quasitriviality of all deformations of (3). These results suggest that quasitriviality could hold true for any deformation (2) of a semisimple bihamiltonian structure of hydrodynamic type.

In this paper we will restrict ourselves to study properties of quasitrivial deformations and leave the discussion on the validity of quasitriviality for any deformation of a semisimple bihamiltonian structure of hydrodynamic type to subsequent publications. The main result of the paper is contained in the following two theorems.

**Theorem 1.** *Any two quasitrivial deformations of a semisimple bihamiltonian structure of hydrodynamic type are equivalent if and only if they are equivalent at the approximation up to  $\epsilon^2$ .*

The property of semisimplicity and the results of [17,28] imply the existence of a local coordinate system near any generic point of  $M$  under which the bihamiltonian structure of hydrodynamic type has the expressions (42)–(44), we call them the canonical coordinates of the semisimple bihamiltonian structure.

**Theorem 2.** *At the approximation up to  $\epsilon^2$ , the space of the equivalence classes of all quasitrivial deformations of a semisimple bihamiltonian structure of hydrodynamic type is parameterized by  $n$  smooth functions  $c_1(u^1), \dots, c_n(u^n)$  of its canonical coordinates.*

We will prove the above theorems by classifying the infinitesimal quasitrivial deformations of a given semisimple bihamiltonian structure of hydrodynamic type, it amounts

to the calculation of certain modification of the second bihamiltonian cohomology. The notion of bihamiltonian cohomology was introduced in [14], it provides an efficient tool to study deformations of bihamiltonian structures of hydrodynamic type. As a direct consequence of the calculation that will be performed in Section 4, we have the following corollary.

**Corollary 1.** *The equivalence classes of infinitesimal quasitrivial deformations of a semisimple bihamiltonian structure of hydrodynamic type are parameterized by  $n$  arbitrary functions of one variable.*

The above results will be understood as to be valid locally near any generic point of the manifold  $M$ . We will first recall the notions of Poisson cohomology and bihamiltonian cohomology in Sections 2 and 3 respectively, and then give the proof of the main results in Section 4, some examples will be given in Section 5.

## 2. Local Poisson structures and Poisson cohomologies

Let  $U$  be a connected open subset of  $\mathbb{R}^n$ . In this section we recall briefly the definition of local Poisson structures and Poisson cohomologies that was presented in [14] on the loop space  $\mathcal{L}(U) = \{\text{smooth maps from } S^1 \text{ to } U\}$  which is treated formally in the spirit of formal variational calculus of [6,8]. For detailed exposition of the following notations see the second chapter of [14] and references therein. Let  $u^1, \dots, u^n$  be a coordinate system on  $U$ , denote by  $\mathcal{A}$  the ring of differential polynomials of the form

$$f(u, u_x, \dots) = \sum_{i_1, s_1, \dots, i_m, s_m} f_{i_1, s_1; \dots; i_m, s_m}(u) u^{i_1, s_1} \dots u^{i_m, s_m}.$$

Here  $u = (u^1, \dots, u^n)$ ,  $u^{(s)} = (u^{1, s}, \dots, u^{n, s})$  with  $u^{i, s} = \partial_x^s u^i(x)$ , and the coefficients of these differential polynomials are smooth functions on  $U$ . Denote

$$\mathcal{A}_0 = \mathcal{A}/\mathbb{R}, \quad \mathcal{A}_1 = \mathcal{A}_0 \, dx, \quad \Lambda = \mathcal{A}_1/d\mathcal{A}_0,$$

where the operator  $d : \mathcal{A}_0 \rightarrow \mathcal{A}_1$  is defined by

$$f \mapsto df = \left( \sum \frac{\partial f}{\partial u^{i, s}} u^{i, s+1} \right) dx.$$

Elements of  $\Lambda$  are called local functionals on the formal loop space, they will be expressed symbolically as

$$\bar{f} = \int f(u(x), u_x(x), \dots, u^{(N)}(x)) dx. \tag{6}$$

In particular, we have  $\int dg = 0$  for any function  $g(u, u_x, \dots, u^{(m)})$  on the formal loop space. Later in Section 4 we will also use functionals of the above form with densities  $f$  being smooth functions of their arguments instead of being differential polynomials.

A local  $k$ -vector on the formal loop space is defined to be a formal infinite sum of the following form:

$$\alpha = \sum \frac{1}{k!} \partial_{x_1}^{s_1} \dots \partial_{x_k}^{s_k} A^{i_1 \dots i_k} \frac{\partial}{\partial u^{i_1, s_1}(x_1)} \wedge \dots \wedge \frac{\partial}{\partial u^{i_k, s_k}(x_k)} \tag{7}$$

with the coefficients  $A$ 's having the expression

$$A^{i_1 \dots i_k} = \sum_{p_2, \dots, p_k \geq 0} B_{p_2 \dots p_k}^{i_1 \dots i_k}(u(x_1), u_x(x_1), \dots) \delta^{(p_2)}(x_1 - x_2) \dots \delta^{(p_k)}(x_1 - x_k). \tag{8}$$

Here  $B_{p_2 \dots p_k}^{i_1 \dots i_k}(u(x_1), u_x(x_1), \dots) \in \mathcal{A}$ , and

$$A^{i_1 \dots i_k} = A^{i_1 \dots i_k}(x_1, \dots, x_k; u(x_1), \dots, u(x_k), \dots) \tag{9}$$

are antisymmetric with respect to the simultaneous permutations  $i_p, x_p \leftrightarrow i_q, x_q$ . These coefficients  $A^{i_1 \dots i_k}$  are called the components of the local  $k$ -vector  $\alpha$ . The space of all such local  $k$ -vectors is denoted by  $\Lambda_{loc}^k$ . In particular, a local vector field on the formal loop space has the form

$$\xi = \sum_{i=1}^n \sum_{s \geq 0} \partial_x^s \xi^i(u(x); u_x(x), \dots) \frac{\partial}{\partial u^{i, s}(x)} \tag{10}$$

which is also called a translation (along  $x$ ) invariant evolutionary vector field. A local bivector takes the form

$$\omega = \frac{1}{2} \sum \partial_x^s \partial_y^t \omega^{ij} \frac{\partial}{\partial u^{i, s}(x)} \wedge \frac{\partial}{\partial u^{j, t}(y)} \tag{11}$$

with

$$\omega^{ij} = A^{ij}(x - y; u(x), u_x(x), \dots) = \sum_{k \geq 0} A_k^{ij}(u(x); u_x(x), \dots) \delta^{(k)}(x - y). \tag{12}$$

The space  $\Lambda_{loc}^0$  is identified with the space of local functionals.

On the space of local multi-vectors

$$\Lambda_{loc}^* = \Lambda_{loc}^0 \oplus \Lambda_{loc}^1 \oplus \Lambda_{loc}^2 \oplus \dots \tag{13}$$

there is defined a bilinear operation of Schouten–Nijenhuis bracket

$$[\cdot, \cdot] : \Lambda_{loc}^k \times \Lambda_{loc}^l \rightarrow \Lambda_{loc}^{k+l-1}, \quad k, l \geq 0. \tag{14}$$

By definition, the Schouten–Nijenhuis bracket of any two elements of  $\Lambda_{loc}^0$  is equal to zero, and that of a local vector field  $\xi$  of the form (10) with a local functional  $f$  of the form (6) is defined by

$$[\xi, \bar{f}] = \int \sum (\partial_x^s \xi^i) \frac{\partial f(u(x), u_x(x), \dots)}{\partial u^{i, s}} dx = \int \sum_{i=1}^n \xi^i \frac{\delta \bar{f}}{\delta u^i(x)} dx, \tag{15}$$

where the variational derivatives are defined by

$$\frac{\delta \bar{f}}{\delta u^i(x)} = \sum_{s \geq 0} (-1)^s \partial_x^s \left( \frac{\partial f}{\partial u^{i,s}} \right), \quad i = 1, \dots, n. \tag{16}$$

The Schouten–Nijenhuis bracket of two local vector fields is given by their usual commutator

$$[\xi, \eta] = \sum \left( \xi^{j,t} \frac{\partial \eta^{i,s}}{\partial u^{j,t}} - \eta^{j,t} \frac{\partial \xi^{i,s}}{\partial u^{j,t}} \right) \frac{\partial}{\partial u^{i,s}} = \sum \partial_x^s \left( \xi^{j,t} \frac{\partial \eta^i}{\partial u^{j,t}} - \eta^{j,t} \frac{\partial \xi^i}{\partial u^{j,t}} \right) \frac{\partial}{\partial u^{i,s}} \tag{17}$$

and components of the Schouten–Nijenhuis bracket of a bivector  $\omega$  of the forms (11) and (12) with a local functional  $\bar{f}$  and with a local vector field  $\xi$  of the form (10) are given respectively by

$$[\omega, \bar{f}]^i = \sum_{j,k} A_k^{ij} \partial_x^k \frac{\delta \bar{f}}{\delta u^j(x)}, \tag{18}$$

$$[\omega, \xi]^{ij} = \sum_{k,t} \left( \partial_x^t \xi^k(u(x); \dots) \frac{\partial A^{ij}}{\partial u^{k,t}(x)} - \frac{\partial \xi^i(u(x); \dots)}{\partial u^{k,t}(x)} \partial_x^t A^{kj} - \frac{\partial \xi^j(u(y); \dots)}{\partial u^{k,t}(y)} \partial_y^t A^{ik} \right). \tag{19}$$

The Schouten–Nijenhuis bracket satisfies the following antisymmetry property and the graded Jacobi identity:

$$[a, b] = (-1)^{kl} [b, a], \tag{20}$$

$$(-1)^{km} [[a, b], c] + (-1)^{kl} [[b, c], a] + (-1)^{lm} [[c, a], b] = 0 \tag{21}$$

for any multi-vectors  $a \in A_{loc}^k, b \in A_{loc}^l, c \in A_{loc}^m$ .

**Definition 1** (Dubrovin and Zhang [14]). A local bivector  $\omega \in A_{loc}^2$  of the form (11) is called a local Poisson structure on the formal loop space  $\mathcal{L}(U)$  if  $[\omega, \omega] = 0$ .

A local Poisson structure given by a bivector of the form (11) can be represented by the Poisson bracket of local functionals

$$\{\bar{f}_1, \bar{f}_2\} = \int \sum_{k \geq 0} \frac{\delta \bar{f}_1}{\delta u^i(x)} A_k^{ij}(u(x); u_x(x), \dots) \partial_x^k \frac{\delta \bar{f}_2}{\delta u^j(x)} dx. \tag{22}$$

It also has the following symbolic representation:

$$\{u^i(x), u^j(y)\} = \sum_{k \geq 0} A_k^{ij}(u(x); u_x(x), \dots) \delta^{(k)}(x - y). \tag{23}$$

There is a natural gradation on the space  $\mathcal{A}$  of differential polynomials

$$\deg u^{i,m} = m, \quad m \geq 1, \quad \deg f(u) = 0. \tag{24}$$

It extends to a gradation on the space of local multi-vectors through the following definition:

$$\deg \frac{\partial}{\partial x} = 1, \quad \deg \frac{\partial}{\partial u^{i,s}} = -s, \quad \deg \delta^{(s)}(x - y) = s + 1. \tag{25}$$

To separate monomials of different degree in a local multi-vector, we introduce a formal indeterminate  $\epsilon$  and assign to it the degree  $-1$ . Denote

$$\Lambda_m^k = \{a \in \Lambda_{\text{loc}}^k \mid \deg a = m\}, \quad \hat{\Lambda}^k = \{a \in \Lambda_{\text{loc}}^k \otimes \mathbb{C}[[\epsilon], \epsilon^{-1}] \mid \deg a = k\}. \tag{26}$$

For a homogeneous element  $f(u, u_x, \dots, u^{(m)})$  of  $\mathcal{A}_0$  we define the degree of the corresponding element  $\bar{f} \in \Lambda_{\text{loc}}^0$  by  $\deg \bar{f} = \deg f - 1$ . For example, an element of  $\hat{\Lambda}^0$  has the form

$$\bar{f} = \int \left( \epsilon^{-1} f_0(u(x)) + \sum_{k=1}^n f_{1,k}(u(x)) u_x^k + \dots \right) dx. \tag{27}$$

The components of a vector field  $\xi \in \hat{\Lambda}^1$  has the form

$$\xi^i = \epsilon^{-1} a^i(u) + \sum_{k=1}^n b_k^i(u) u_x^k + \epsilon \left( \sum_{k=1}^n c_k^i(u) u_{xx}^k + \sum_{k,l=1}^n e_{kl}^i(u) u_x^k u_x^l \right) + \dots. \tag{28}$$

A Poisson structure  $\omega \in \Lambda_2^2$  is of hydrodynamic type and has the representation of the form (1), any Poisson structure of the form  $\omega + P(\epsilon) \in \hat{\Lambda}^2$  with  $P(\epsilon) = \sum_{k \geq 1} \epsilon^k P_k, P_k \in \Lambda_{k+2}^2$  is called a deformation of  $\omega$ .

The space

$$\hat{\Lambda} = \hat{\Lambda}^0 \oplus \hat{\Lambda}^1 \oplus \hat{\Lambda}^2 \oplus \dots \tag{29}$$

is closed with respect to the Schouten–Nijenhuis bracket  $\epsilon[\cdot, \cdot]$ , and a Poisson structure  $\omega \in \hat{\Lambda}^2$  defines a differential

$$\epsilon \partial : \hat{\Lambda}^k \rightarrow \hat{\Lambda}^{k+1}, \quad \epsilon \partial a = \epsilon[\omega, a], \quad a \in \hat{\Lambda}^k. \tag{30}$$

The cohomology of the complex  $(\hat{\Lambda}, \epsilon \partial)$  is called the Poisson cohomology of the Poisson structure  $\omega$ , and is denoted by  $H^*(\mathcal{L}(U), \omega)$ . It is a natural generalization of the notion of Poisson cohomology for finite dimensional Poisson structures [24].

### 3. Bihamiltonian structures and bihamiltonian cohomologies

Assume that on the formal loop space  $\mathcal{L}(U)$  of a ball  $U$  in  $\mathbb{R}^n$  we are given two Poisson structures  $\omega_1, \omega_2$  of hydrodynamic type with components of the form

$$\omega_a^{ij} = g_a^{ij}(u) \delta'(x - y) + \Gamma_{k,a}^{ij}(u) u_x^k \delta(x - y), \quad \det(g_a^{ij}) \neq 0, \quad a = 1, 2. \tag{31}$$

If the linear combination  $\omega_\lambda = \omega_2 - \lambda \omega_1$  is also a Poisson structure of the above form for an arbitrary parameter  $\lambda \in \mathbb{R}$ , then the pair  $(\omega_1, \omega_2)$  is called a bihamiltonian structure of hydrodynamic type. These two Poisson structures define two complexes  $(\hat{\Lambda}, \epsilon \partial_a), a = 1, 2$ . It is proved in [7,22] that the Poisson cohomologies  $H^*(\mathcal{L}(U), \omega_a), a = 1, 2$  are trivial (also

see [14] for a different proof of triviality for the first and the second Poisson cohomologies). Thus any deformation  $\omega_a + P(\epsilon) \in \hat{\Lambda}^2$  of a single Poisson structure  $\omega_a$  can be obtained from  $\omega_a$  by performing a Miura transformation of the form (4). Instead of the deformations of a single Hamiltonian structure, we are interested in deformations of the bihamiltonian structure  $(\omega_1, \omega_2)$ . Due to the triviality of the Poisson cohomology  $H^*(\mathcal{L}(U), \omega_1)$ , we can always assume that our deformations keep the first Poisson structure  $\omega_1$  unchanged.

**Definition 2.** The pair of bivectors

$$\left( \omega_1, \omega_2 + \sum_{m \geq 1} \epsilon^m P_m \right), \quad P_m \in \Lambda_{m+2}^2 \tag{32}$$

is called a deformation of the bihamiltonian structure  $(\omega_1, \omega_2)$  if the equality

$$\left[ \omega_2 + \sum_{m \geq 1} \epsilon^m P_m - \lambda \omega_1, \omega_2 + \sum_{m \geq 1} \epsilon^m P_m - \lambda \omega_1 \right] = 0 \tag{33}$$

holds true for an arbitrary parameter  $\lambda$ . It is called an  $N$ th order deformation of the bihamiltonian structure  $(\omega_1, \omega_2)$  if the equality (33) holds true for an arbitrary parameter  $\lambda$  at the approximation up to  $\epsilon^N$ .

**Definition 3.** We say that two deformations (of order  $N$ ) of the bihamiltonian structure  $(\omega_1, \omega_2)$  are equivalent or quasi-equivalent if they are related (resp. at the approximation up to  $\epsilon^N$ ) by a Miura transformation (4) or by a quasi-Miura transformation (5). A deformation (of order  $N$ ) of the bihamiltonian structure  $(\omega_1, \omega_2)$  is called trivial or quasitrivial if it is equivalent or quasi-equivalent to  $(\omega_1, \omega_2)$  (resp. at the approximation up to  $\epsilon^N$ ).

Due to the above definition, for a  $N$ th order deformation (32) the bivectors  $P_m$  must satisfy the conditions

$$\partial_1 P_m = 0, \quad 1 \leq m \leq N, \tag{34}$$

$$\partial_2 P_1 = 0, \quad 2\partial_2 P_m + \sum_{k=1}^{m-1} [P_k, P_{m-k}] = 0, \quad 2 \leq m \leq N. \tag{35}$$

Here the differentials  $\partial_1, \partial_2$  are defined by the Poisson structures  $\omega_1$  and  $\omega_2$  respectively as in (30), they act on the subspaces  $\Lambda_m^k$  as

$$\partial_a : \Lambda_m^k \rightarrow \Lambda_{m+2}^{k+1}, \quad k \geq 0, \quad m \geq k - 1, \quad a = 1, 2. \tag{36}$$

The notion of bihamiltonian cohomologies  $H^k = \oplus_{m \geq k-1} H_m^k, k \geq 0$  for  $(\omega_1, \omega_2)$  is introduced in [14], they are defined by

$$\begin{aligned} H_m^k(\mathcal{L}(U); \omega_1, \omega_2) &= \text{Ker}(\partial_1 \partial_2|_{\Lambda_m^{k-1}}) / \text{Im}(\partial_1|_{\Lambda_m^{k-2}}) \oplus \text{Im}(\partial_2|_{\Lambda_m^{k-2}}), \quad k \geq 2, \\ H_m^1(\mathcal{L}(U); \omega_1, \omega_2) &= \text{Ker}(\partial_1 \partial_2|_{\Lambda_m^0}), \\ H_m^0(\mathcal{L}(U); \omega_1, \omega_2) &= \text{Ker}(\partial_1|_{\Lambda_m^0}) \cap \text{Ker}(\partial_2|_{\Lambda_m^0}). \end{aligned} \tag{37}$$



It was proved in [14] that the zeroth cohomologies  $H_m^0(\mathcal{L}(U); \omega_1, \omega_2)$  correspond to the space of common Casimirs of the Poisson structures  $\omega_1, \omega_2$ , the first cohomologies  $H_m^1(\mathcal{L}(U); \omega_1, \omega_2)$  correspond to the space of bihamiltonian vector fields, and the second cohomologies  $H_m^2(\mathcal{L}(U); \omega_1, \omega_2)$  correspond to the space of infinitesimal deformations of the bihamiltonian structure modulo the trivial deformations caused by Miura transformations. Below we list some other simple propositions on the second and third cohomologies.

**Proposition 1.**

- (1) The bihamiltonian cohomologies  $H_i^2(\mathcal{L}(U); \omega_1, \omega_2)$  vanish for  $K + 1 \leq i \leq N$  iff any class of deformations of the bihamiltonian structure  $(\omega_1, \omega_2)$  of order  $s \leq N$  is uniquely determined by the corresponding class of deformations of order  $K$ .
- (2) The bihamiltonian cohomologies  $H_{2k+1}^2(\mathcal{L}(U); \omega_1, \omega_2)$  vanish for  $1 \leq 2k + 1 \leq N$  iff any deformation of the bihamiltonian structure  $(\omega_1, \omega_2)$  is equivalent to a deformation of the form (32) with  $P_{2l+1} = 0, 2l + 1 \leq N$ .

**Proof.** Let us first assume that  $H_i^2(\mathcal{L}(U); \omega_1, \omega_2)$  vanishes for  $K + 1 \leq i \leq N$ . We need to prove that any two deformations of order  $s \leq N$  of the form

$$\left( \omega_1, \omega_2 + \sum_{m=1}^K \epsilon^m P_m + \sum_{m=K+1}^s \epsilon^m P_m^{(l)} \right) + \mathcal{O}(\epsilon^{s+1}), \quad l = 1, 2 \tag{38}$$

are equivalent. By using the identities in (34) and (35) we can find  $X, Y \in \Lambda_{K+1}^1$  such that

$$P_{K+1}^{(1)} = \partial_1 X, \quad P_{K+1}^{(2)} = \partial_1 Y.$$

From (35) it follows that

$$\partial_2 \partial_1 (X - Y) = 0.$$

So our assumption implies the existence of  $I, J \in \Lambda_{K-1}^0$  such that

$$X = Y + \partial_1 I + \partial_2 J.$$

Thus after the Miura transformation

$$u^i \mapsto \exp(-\epsilon^{K+1} \partial_1 J) u^i$$

the first deformed bihamiltonian structure

$$\left( \omega_1, \omega_2 + \sum_{m=1}^K \epsilon^m P_m + \sum_{m=K+1}^s \epsilon^m P_m^{(1)} \right) + \mathcal{O}(\epsilon^{s+1})$$

is transformed to

$$\left( \omega_1, \omega_2 + \sum_{m=1}^K \epsilon^m P_m + \epsilon^{K+1} P_{K+1}^{(2)} + \sum_{m=K+2}^s \epsilon^m \tilde{P}_m^{(1)} \right) + \mathcal{O}(\epsilon^{s+1}).$$

By repeating the same procedure, we prove the equivalence of the two deformations of (38).

Now we assume that any class of deformations of the bihamiltonian structure  $(\omega_1, \omega_2)$  of order  $s \leq N$  is uniquely determined by the corresponding class of deformations of order  $K$ . For any

$$X \in \text{Ker}(\partial_1 \partial_2 |_{\Lambda_s^1}), \quad K + 1 \leq s \leq N$$

we have a  $s$ th order deformation of the form

$$(\omega_1, \omega_2 + \epsilon^s \partial_1 X). \tag{39}$$

It follows from our assumption that there exists a Miura type transformation

$$u^i \mapsto u^i + \sum_{j=1}^s \epsilon^j A_j^i, \quad A_j \in \Lambda_j^1$$

that transforms the bihamiltonian structure  $(\omega_1, \omega_2)$  to (39), i.e.,

$$\begin{aligned} \omega_1 &= e^{-\epsilon^s \text{ad}_{\tilde{A}_s}} \dots e^{-\epsilon \text{ad}_{\tilde{A}_1}} \omega_1 + \mathcal{O}(\epsilon^{s+1}), \\ \omega_2 + \epsilon^s \partial_1 X &= e^{-\epsilon^s \text{ad}_{\tilde{A}_s}} \dots e^{-\epsilon \text{ad}_{\tilde{A}_1}} \omega_2 + \mathcal{O}(\epsilon^{s+1}). \end{aligned} \tag{40}$$

Here we represent, modulo  $\epsilon^{s+1}$ , the Miura transformation as the composition of the one parameter transformation groups  $u \mapsto e^{\epsilon^k \tilde{A}_k} u$ ,  $k = 1, \dots, s$  corresponding to the vector fields

$$\tilde{A}_1^i = A_1^i, \quad \tilde{A}_2^i = A_2^i - \frac{1}{2} \sum_{j=1}^n \sum_{t \geq 0} \frac{\partial A_1^i}{\partial u^{j,t}} \partial_x^t A_1^j, \dots$$

From the identities in (40) we obtain

$$\partial_1 \tilde{A}_s = 0, \quad \partial_2 \tilde{A}_s = \partial_1 X.$$

The first equality yields the existence of  $I \in \Lambda_{s-2}^0$  such that  $\tilde{A}_s = \partial_1 I$ , and from the second equality it follows that  $X \in \text{Im}(\partial_1 |_{\Lambda_{s-2}^0}) \oplus \text{Im}(\partial_2 |_{\Lambda_{s-2}^0})$ . Thus we proved the first part of the proposition. The second part can be proved in a similar way. The proposition is proved.  $\square$

**Proposition 2.** *If the bihamiltonian cohomology  $H_{N+3}^3(\mathcal{L}(U); \omega_1, \omega_2)$  vanishes then any  $N$ th order deformation of the bihamiltonian structure  $(\omega_1, \omega_2)$  can be extended to an  $N + 1$ th order deformation.*

**Proof.** Any  $N$ th order deformation can be represented as

$$\left( \omega_1, \omega_2 + \sum_{i=1}^N \epsilon^i \partial_1 X_i \right) + \mathcal{O}(\epsilon^{N+1}), \quad X_i \in \Lambda_i^1.$$

In order to extend it to a deformation of order  $N + 1$  we need to find a local vector field  $X_{N+1} \in \Lambda_{N+1}^1$  such that

$$\partial_1 \partial_2 X_{N+1} = \frac{1}{2} \sum_{i=1}^N [\partial_1 X_i, \partial_1 X_{N+1-i}]. \quad (41)$$

Denote by  $Q$  the r.h.s. of the above equation. Then by using the graded Jacobi identity (21) of the Schouten–Nijenhuis bracket and the equalities

$$\partial_1 \partial_2 X_m = \frac{1}{2} \sum_{i=1}^{m-1} [\partial_1 X_i, \partial_1 X_{m-i}], \quad m = 1, \dots, N$$

we obtain

$$\partial_1 Q = \partial_2 Q = 0.$$

So there exists  $R \in \Lambda_{N+3}^2$  such that  $Q = \partial_1 R$ . Now it follows from the equality  $\partial_1 \partial_2 R = 0$  and our assumption of the proposition that

$$R = \partial_1 A + \partial_2 B, \quad A, B \in \Lambda_{N+1}^1.$$

So Eq. (41) now takes the form

$$\partial_1 \partial_2 X_{N+1} = \partial_1 (\partial_1 A + \partial_2 B)$$

and it has a solution  $X_{N+1} = B$ . The proposition is proved.  $\square$

Due to the above propositions, the problem of classification of deformations of the hydrodynamic bihamiltonian structures is reduced to the computation of bihamiltonian cohomologies. We can also consider certain modification of the bihamiltonian cohomology in order to deal with quasitrivial deformations of the hydrodynamic bihamiltonian structures, we will do this in the next section.

#### 4. Computation of a modified bihamiltonian cohomology and the proof of the main theorems

We consider in this section the problem of classification of infinitesimal quasitrivial deformations of a semisimple bihamiltonian structure  $(\omega_1, \omega_2)$  on the formal loop space of

$U$  with components of the form (31). Here  $U$  is defined as in Section 3. Since our study is in a local nature, we may adjust  $U$  to a sufficiently small neighborhood of a generic point of the bihamiltonian structure to ensure the existence on  $U$  a coordinate system  $u^1, \dots, u^n$ , called the canonical coordinates of the semisimple bihamiltonian structure, such that both metrics  $g_1^{ij}$  and  $g_2^{ij}$  are diagonal under these coordinates, and the identities  $g_2^{ii} = u^i g_1^{ii}$  hold true [17,28]. In terms of these coordinates the bihamiltonian structure can be expressed as

$$\omega_1^{ij} = f^i \delta^{ij} \delta'(x - y) + \frac{1}{2} f_x^i \delta^{ij} \delta(x - y) + A^{ij} \delta(x - y), \tag{42}$$

$$\omega_2^{ij} = g^i \delta^{ij} \delta'(x - y) + \frac{1}{2} g_x^i \delta^{ij} \delta(x - y) + B^{ij} \delta(x - y). \tag{43}$$

Here  $f^i = f^i(u^1, \dots, u^n)$ ,  $g^i = u^i f^i$ ,  $f_x^i = \partial_x f^i$ ,  $g_x^i = \partial_x g^i$ , and

$$A^{ij} = \frac{1}{2} \left( \frac{f^i}{f^j} f_i^j u_x^j - \frac{f^j}{f^i} f_j^i u_x^i \right), \quad B^{ij} = \frac{1}{2} \left( \frac{u^i f^i}{f^j} f_i^j u_x^j - \frac{u^j f^j}{f^i} f_j^i u_x^i \right), \tag{44}$$

where  $f_b^a = \frac{\partial f^a}{\partial u^b}$ .

Denote by  $\Omega$  the space of local functionals of the form

$$\bar{f} = \int f(u, u_x, \dots, u^{(m)}) dx,$$

where  $f$  is a smooth function of all of its arguments. Define

$$\begin{aligned} \hat{H}^2(\mathcal{L}(U); \omega_1, \omega_2) &= \bigoplus_{m \geq 1} \hat{H}_m^2, \\ \hat{H}_m^2 &= H_m^2(\mathcal{L}(U); \omega_1, \omega_2) \cap (\partial_1 \Omega \oplus \partial_2 \Omega). \end{aligned} \tag{45}$$

Then  $\hat{H}^2$  is the space of equivalence classes of infinitesimal quasitrivial deformations of the bihamiltonian structure  $(\omega_1, \omega_2)$ . In fact, an infinitesimal deformation of the bihamiltonian structure  $(\omega_1, \omega_2)$  can be represented in the form  $(\omega_1, \omega_2 + \epsilon \partial_1 X)$  by a local vector field  $X$  satisfying  $\partial_1 \partial_2 X = 0$ . Quasitriviality implies the existence of a vector field  $Y$  with components  $Y^i$  that depend rationally on the jet coordinates  $u^{(m)}$ ,  $m \geq 1$  such that the infinitesimal deformation is obtained from the original bihamiltonian structure by the infinitesimal quasi-Miura transformation

$$u^i \mapsto u^i - \epsilon Y^i, \quad i = 1, \dots, n. \tag{46}$$

So the vector field  $Y$  must satisfy  $\partial_1 Y = 0$ ,  $\partial_2 Y = \partial_1 X$ , and consequently the vector field  $X$  can be represented as  $\partial_2 I - \partial_1 J$ ,  $I, J \in \Omega$ . Conversely, from the result of the following theorem we know that any element of  $\hat{H}^2$  represents an equivalence class of quasitrivial infinitesimal deformation of the bihamiltonian structure.

**Theorem 3.** We have  $\hat{H}_m^2 = 0$  for  $m = 1, 3, 4, \dots$  and

$$\hat{H}_2^2 = \left\{ \sum_{i=1}^n \left( \partial_1 \int (u^i c_i(u^i) u_x^i \log u_x^i) dx - \partial_2 \int (c_i(u^i) u_x^i \log u_x^i) dx \right) \right\}. \tag{47}$$

Here  $\partial_1, \partial_2$  are the differentials defined by the Poisson structures  $\omega_1$  and  $\omega_2$  respectively,  $c_i(u^i)$  are arbitrary smooth functions of  $u^i$ . Moreover, two sets of functions  $\{c_i\}$  and  $\{\tilde{c}_i\}$  define the same element in  $\hat{H}^2$  iff  $c_i = \tilde{c}_i$ .

We will use the symbol

$$A(u, u_x, \dots, u^{(N)}) \sim B(u, u_x, \dots, u^{(N)})$$

to indicate that the difference of the functions  $A$  and  $B$  is a differential polynomial. In order to prove the above theorem we first need to prove some lemmas.

**Lemma 1.** Let  $X = \partial_2 I - \partial_1 J \in \hat{H}^2$  with

$$I = \int G(u, u_x, \dots, u^{(N)}) dx, \quad J = \int H(u, u_x, \dots, u^{(N)}) dx, \quad N \geq 2.$$

Then the densities  $G, H$  can be chosen to have the form

$$G \sim \sum_{i=1}^n \frac{(u^{i,N})^2}{u_x^i} P^i(u; u_x, \dots, u^{(N-2)}; u^{i,N-1}) + Q(u, \dots, u^{(N-1)}), \tag{48}$$

$$H \sim \sum_{i=1}^n \frac{(u^{i,N})^2}{u_x^i} u^i P^i(u; u_x, \dots, u^{(N-2)}; u^{i,N-1}) + R(u, \dots, u^{(N-1)}). \tag{49}$$

Here  $P^i(u; u_x, \dots, u^{(N-2)}; u^{i,N-1})$  are differential polynomials,  $Q, R$  are smooth functions, and any nonzero differential polynomial  $P^i(u; u_x, \dots, u^{(N-2)}; u^{i,N-1})$  is indivisible by  $u_x^i$ .

**Proof.** Denote by  $X^i, i = 1, \dots, n$  the components of the local vector field  $X$ , from our assumption we know that they are differential polynomials. We are to use this property repeatedly to prove the lemma. Let us start with the polynomiality of  $\frac{\partial X^i}{\partial u^{i,2N+1}}$ . Denote

$$X_{j,m}^i = \frac{\partial X^i}{\partial u^{j,m}}, \quad G_{i,p;j,q} = \frac{\partial^2 G}{\partial u^{i,p} \partial u^{j,q}}, \quad H_{i,p;j,q} = \frac{\partial^2 H}{\partial u^{i,p} \partial u^{j,q}}.$$

By using the simple identity

$$\frac{\partial}{\partial u^{i,k}} \partial_x^m = \sum_{l=0}^m \binom{m}{l} \partial_x^l \frac{\partial}{\partial u^{i,k-m+l}}$$

and the forms (42) and (43) of the bihamiltonian structure  $(\omega_1, \omega_2)$  we obtain the following formulae:

$$(-1)^N X_{j,2N+1}^i = g^i G_{i,N;j,N} - f^i H_{i,N;j,N}. \tag{50}$$

It follows that the functions  $G$  and  $H$  satisfy the relations

$$u^i G_{i,N;j,N} - H_{i,N;j,N} \sim 0, \quad (u^i - u^j) G_{i,N;j,N} \sim 0. \tag{51}$$

So there exist smooth functions  $a_i, b_i, c$ , such that

$$G \sim \sum_{i=1}^n a_i(u, \dots, u^{(N-1)}, u^{i,N}),$$

$$H \sim \sum_{i=1}^n (u^i a_i(u, \dots, u^{(N-1)}, u^{i,N}) + b_i(u, \dots, u^{(N-1)})u^{i,N}) + c(u, \dots, u^{(N-1)}).$$

By substituting these expressions into the relations  $(-1)^N \frac{\partial X^i}{\partial u^{i,2N}} \sim 0$  we obtain

$$-\left(N + \frac{1}{2}\right) f^i u_x^i \frac{\partial^2 a_i}{\partial u^{i,N} \partial u^{i,N}} \sim 0.$$

Thus we can find differential polynomials  $p_i(u, \dots, u^{(N-1)}, u^{i,N})$  and smooth functions  $q_i(u, \dots, u^{(N-1)}), r_i(u, \dots, u^{(N-1)})$  such that

$$a_i = \frac{p_i(u, \dots, u^{(N-1)}, u^{i,N})}{u_x^i} + q_i(u, \dots, u^{(N-1)})u^{i,N} + r_i(u, \dots, u^{(N-1)}).$$

Now the functions  $G, H$  can be written in the form

$$G \sim \sum_{i=1}^n \left( \frac{p_i(u, \dots, u^{(N-1)}, u^{i,N})}{u_x^i} + q_i(u, \dots, u^{(N-1)})u^{i,N} \right) + r(u, \dots, u^{(N-1)}), \tag{52}$$

$$H \sim \sum_{i=1}^n \left( u^i \frac{p_i(u, \dots, u^{(N-1)}, u^{i,N})}{u_x^i} + s_i(u, \dots, u^{(N-1)})u^{i,N} \right) + e(u, \dots, u^{(N-1)}). \tag{53}$$

Here  $s_i, e$  are some smooth functions. In the above expression of  $G, H$ , we assume that the differential polynomials  $p_i$  do not contain terms that are linear and constant with respect to  $u^{i,N}$ , such terms can be absorbed into the functions  $q_i u^{i,N}, s_i u^{i,N}$  and  $r, e$ .

Assuming the forms (52) and (53) of the functions  $G, H$  we continue to use the polynomiality of  $(-1)^N \frac{\partial X^i}{\partial u^{j,2N}}$  with  $i \neq j$  to obtain

$$u^i (G_{i,N;j,N-1} - G_{j,N;i,N-1}) - (H_{i,N;j,N-1} - H_{j,N;i,N-1}) \sim 0.$$

From these relations it follows that for indices  $i \neq j$  we have

$$H_{i,N;j,N-1} - H_{j,N;i,N-1} \sim 0, \quad G_{i,N;j,N-1} - G_{j,N;i,N-1} \sim 0, \tag{54}$$

$$G_{i,N;i,N;j,N-1} - G_{i,N;i,N-1;j,N} \sim G_{i,N;i,N;j,N-1} \sim 0. \tag{55}$$

The relation (55) shows that we can adjust the differential polynomials  $p_i$  so that they have the form

$$p_i = p_i(u, \dots, u^{(N-2)}, u^{i,N-1}, u^{i,N}), \quad i = 1, \dots, n.$$

Now by substituting the expression (53) for the function  $H$  into the first relation of (54) we arrive at

$$\frac{\partial s_i}{\partial u^{j,N-1}} - \frac{\partial s_j}{\partial u^{i,N-1}} \sim 0,$$

by using the Poincaré lemma we can find differential polynomials  $\hat{s}_1, \dots, \hat{s}_n$  such that the identity

$$\frac{\partial(s_i - \hat{s}_i)}{\partial u^{j,N-1}} - \frac{\partial(s_j - \hat{s}_j)}{\partial u^{i,N-1}} = 0$$

hold true. This identity implies the existence of a function  $W(u, \dots, u^{(N-1)})$  satisfying

$$s_i \sim \frac{\partial W}{\partial u^{i,N-1}}, \quad i = 1, \dots, n.$$

So by adjusting the density  $H$  of the functional  $J$  to  $H - \partial_x W$  we can assume that in the expression (53) for the function  $H$  the second term  $\sum_{i=1}^n s_i u^{i,N}$  does not appear. In a similar way, we can also assume that the term  $\sum_{i=1}^n q_i u^{i,N}$  in the expression (52) of the density of the functional  $I$  vanishes.

Finally, the relation  $(-1)^N \frac{\partial^2 X^i}{\partial u^{i,2N-1} \partial u^{i,N}} \sim 0$  implies that

$$\frac{N^2}{2} \frac{f^i u_{xx}^i}{u^i} \frac{\partial^3 p_i}{\partial u^{i,N} \partial u^{i,N} \partial u^{i,N}} \sim 0, \quad i = 1, \dots, n. \tag{56}$$

So we can adjust the densities  $G, H$  of the functionals  $I, J$  so that they have the forms (48) and (49). The lemma is proved.  $\square$

Let us introduce the operators

$$Z_{ij}^m = \sum_{p \geq m} (-1)^p \binom{p}{m} \frac{\partial^2}{\partial u^{i,p} \partial u^{j,2N+m-p}}, \quad 1 \leq i, j \leq n, \quad m \geq 0.$$

It is easy to verify that these operators satisfy the identities  $[\partial_x, Z_{ij}^m] = Z_{ij}^{m-1}$  and, moreover, we have the following lemma.

**Lemma 2.** For a functional  $I = \int G(u, u^{(1)}, \dots) dx$ , denote

$$I_k = \frac{\delta I}{\delta u^k}, \quad k = 1, \dots, n. \tag{57}$$

Then for any indices  $i, j, m$ , the following formulae hold true:

$$Z_{ij}^m I_k = \sum_{s \geq 0} \binom{s+m}{s} (-\partial_x)^s \frac{\partial}{\partial u^{k,s+m}} \left( \frac{\partial I_i}{\partial u^{j,2N}} \right).$$

**Proof.** It is well known from the theory of variational calculus that for any functional  $I$  we have the following identities:

$$\frac{\partial}{\partial u^{i,p}} \left( \frac{\delta I}{\delta u^k} \right) = \sum_{t \geq p} (-1)^t \binom{t}{p} \partial_x^{t-p} \frac{\partial}{\partial u^{k,t}} \left( \frac{\delta I}{\delta u^i} \right).$$

From which it follows that

$$\frac{\partial^2 I_k}{\partial u^{i,p} \partial u^{j,2N+m-p}} = \sum_{s \geq 0} \sum_{t \geq p} (-1)^{s+t} \binom{s+t}{p} \binom{s+t-p}{s} \partial_x^s \frac{\partial^2 I_i}{\partial u^{k,s+t} \partial u^{j,2N+m-t}}.$$

By using this identity we obtain

$$\begin{aligned} Z_{ij}^m I_k &= \sum_{p \geq 0} (-1)^p \binom{p}{m} \frac{\partial^2 I_k}{\partial u^{i,p} \partial u^{j,2N+m-p}} \\ &= \sum_{p \geq 0} (-1)^p \binom{p}{m} \sum_{s \geq 0} \sum_{t \geq p} (-1)^{s+t} \binom{s+t}{p} \binom{s+t-p}{s} \partial_x^s \frac{\partial^2 I_i}{\partial u^{k,s+t} \partial u^{j,2N+m-t}} \\ &= \sum_{s \geq 0} (-\partial_x)^s \sum_{t \geq 0} (-1)^t \binom{s+t}{s} \left[ \sum_{p=0}^t (-1)^p \binom{p}{m} \binom{t}{p} \right] \frac{\partial^2 I_i}{\partial u^{k,s+t} \partial u^{j,2N+m-t}} \\ &= \sum_{s \geq 0} (-\partial_x)^s \binom{s+m}{s} \frac{\partial^2 I_i}{\partial u^{k,s+m} \partial u^{j,2N}}. \end{aligned}$$

Here we assumed  $\binom{p}{m} = 0$  when  $p \leq m - 1$  and we used the identity

$$\sum_{p=0}^t (-1)^p \binom{p}{m} \binom{t}{p} = (-1)^t \delta_{tm}.$$

The lemma is proved.  $\square$

**Lemma 3.** The polynomials  $P^i$  defined in Lemma 1 must vanish.

**Proof.** Let  $m$  be the highest order of the  $x$ -derivatives of  $u^1, \dots, u^n$  that appear in the polynomials  $P^i$ . We first prove, by using the polynomiality of  $Z_{ij}^{m-1} X^k$ , that  $m$  must be less



than 3. To this end, let us assume at the moment that  $m \geq 3$ . From the forms (42) and (43) of the bihamiltonian structure  $(\omega_1, \omega_2)$  we know that the components of the vector field  $X = \partial_2 I - \partial_1 J$  can be expressed as

$$X^k = g^k \partial_x \frac{\delta I}{\delta u^k} + \frac{\partial_x g^k}{2} \frac{\delta I}{\delta u^k} + \sum_{\alpha=1}^n B^{k\alpha} \frac{\delta I}{\delta u^\alpha} - f^k \partial_x \frac{\delta J}{\delta u^k} - \frac{\partial_x f^k}{2} \frac{\delta J}{\delta u^k} - \sum_{\alpha=1}^n A^{k\alpha} \frac{\delta J}{\delta u^\alpha}.$$

Since the highest order of the  $x$ -derivatives of  $u^p$  that appear in  $\frac{\delta I}{\delta u^k}$  is  $2N$ , we have

$$\begin{aligned} Z_{ij}^{m-1} X^k &= g^k (\partial_x Z_{ij}^{m-1} - Z_{ij}^{m-2}) I_k + \frac{\partial_x g^k}{2} Z_{ij}^{m-1} I_k + \sum_{\alpha=1}^n B^{k\alpha} Z_{ij}^{m-1} I_\alpha \\ &\quad - f^k (\partial_x Z_{ij}^{m-1} - Z_{ij}^{m-2}) J_k - \frac{\partial_x f^k}{2} Z_{ij}^{m-1} J_k - \sum_{\alpha=1}^n A^{k\alpha} Z_{ij}^{m-1} J_\alpha. \end{aligned}$$

Here  $I_k, J_k$  are defined as in (49). By using Lemmas 1 and 2 we know that

$$\frac{\partial I_i}{\partial u^{j,2N}} \sim (-1)^N \frac{2P^i}{u_x^i} \delta_{ij}$$

and

$$\begin{aligned} Z_{ij}^{m-1} I_k &\sim \left( \frac{\partial}{\partial u^{k,m-1}} - m \partial_x \frac{\partial}{\partial u^{k,m}} \right) \frac{\partial I_i}{\partial u^{j,2N}}, \\ Z_{ij}^{m-2} I_k &\sim \left( \frac{\partial}{\partial u^{k,m-2}} - (m-1) \partial_x \frac{\partial}{\partial u^{k,m-1}} + \frac{m(m-1)}{2} \partial_x^2 \frac{\partial}{\partial u^{k,m}} \right) \frac{\partial I_i}{\partial u^{j,2N}}. \end{aligned}$$

We can get similar expression for  $Z_{ij}^{m-1} J_k$  and  $Z_{ij}^{m-2} J_k$ . By using these formulae, we see that for the case  $i = j \neq k$  the term with the highest power of  $\frac{1}{u_x^i}$  in the expression of  $Z_{ij}^{m-1} X^k$  is given by

$$(-1)^N 2m(m+1) f^k (u^i - u^k) \frac{(u_{xx}^i)^2}{(u_x^i)^3} \frac{\partial P^i}{\partial u^{k,m}}. \tag{58}$$

From the fact that  $P^i$  is indivisible by  $u_x^i$  and  $Z_{ii}^{m-1} X^k$  is a differential polynomial it follows that  $P^i$  does not depend on  $u^{k,m}$  for  $k \neq i$ . In the case when  $i = j = k$  we have

$$Z_{ij}^{m-1} X^k \sim (-1)^{N+1} m^2 f^i \frac{u_{xx}^i}{u_x^i} \frac{\partial P^i}{\partial u^{i,m}}. \tag{59}$$

So  $P^i$  does not depend on  $u^{i,m}$  either. Thus we proved that the highest order  $m$  of the  $x$ -derivatives of  $u^1, \dots, u^n$  that appear in the polynomial  $P^i$  must be less than 3. To complete the proof of the lemma we use the polynomiality of  $Z_{ij}^1 X^k$ . In the same way as we did above, we can prove that the terms (58) for the case of  $m = 2$  is a differential polynomial, so  $P^i$

does not depend on  $u_{xx}^k$  for  $i \neq k$ . Then the counterpart of (59) for the case of  $m = 2$  has the form

$$Z_{ii}^1 X^i \sim \frac{(-1)^{N+1} f^i}{u_x^i} \left( 4u_{xx}^i \frac{\partial P^i}{\partial u_{xx}^i} + (2N - 2)P^i \right) \tag{60}$$

which implies  $P^i = 0$ . The lemma is proved.  $\square$

Now we can prove the main result of this section.

**Proof of Theorem 3.** By using the above lemma, we know that for any element of  $\hat{H}^2$  we can choose its representative  $X \in \text{Ker}(\partial_1 \partial_2)$  of the form

$$X = \partial_2 I - \partial_1 J, \quad I = \int G(u, u_x) dx, \quad J = \int H(u, u_x) dx. \tag{61}$$

Then the polynomiality of

$$\frac{\partial X^i}{\partial u_x^j} = f^i \frac{\partial^2 H}{\partial u_x^i \partial u_x^j} - g^j \frac{\partial^2 G}{\partial u_x^i \partial u_x^j} \tag{62}$$

allows us to adjust the vector field  $X$  such that the functions  $G$  and  $H$  have the expression

$$G = \sum_{i=1}^n h_i(u^1, \dots, u^n, u_x^i), \quad H = \sum_{i=1}^n u^i h_i(u^1, \dots, u^n, u_x^i). \tag{63}$$

By using the identity

$$\frac{\partial X^i}{\partial u_{xx}^i} = \frac{3}{2} f^i u_x^i \frac{\partial^2 h_i}{\partial u_x^i \partial u_x^i} \tag{64}$$

we see that the functions  $h_i$  must take the form

$$h_i = -c_i(u) u_x^i \log u_x^i + \text{differential polynomial}. \tag{65}$$

Now from the explicit form of  $\frac{\partial X^i}{\partial u_x^j}$  we know that

$$(u^i - u^j) \frac{\partial c_j}{\partial u^i} \log u_x^j \tag{66}$$

are differential polynomials, thus we have  $\frac{\partial c_j}{\partial u^i} = 0$  for  $i \neq j$ , and  $c_i$  depend only on  $u^i$ . So we proved that any element of  $\hat{H}^2$  has a representative of the form given in the right hand side of (47).

On the other hand, given any vector field  $X$  with the form given in the right hand side of (47), we can easily verify that its components have the expressions

$$X^i = - \sum_{j=1}^n \left[ \left( \frac{1}{2} \delta_{ij} \partial_x f^i + A^{ij} \right) c_j u_x^j + (2\delta_{ij} f^i - L^{ij}) \partial_x (c_j u_x^j) \right]. \tag{67}$$

Here

$$L^{ij} = \frac{1}{2} \delta_{ij} f^i + \frac{(u^i - u^j) f^i}{2f^j} \frac{\partial f^j}{\partial u^i}. \tag{68}$$

It shows that  $X^i$  are differential polynomials and thus  $X$  is a representative of an element of  $\hat{H}^2$ .

Finally, we are left to show that a vector field  $X$  of the form given in the right hand side of (47) is trivial if and only if  $c_1 = \dots = c_n = 0$ . From the expression (67) it follows that the triviality of the vector field  $X$  is equivalent to the existence of functions  $\alpha_i(u), \beta_i(u), i = 1, \dots, n$  such that the vector fields  $X$  can be expressed as  $\tilde{X} = \partial_2 \tilde{I} - \partial_1 \tilde{J}$ , where the functionals  $\tilde{I}$  and  $\tilde{J}$  have the form

$$\tilde{I} = \int \sum_{i=1}^n \alpha_i(u) u_x^i dx, \quad \tilde{J} = \int \sum_{i=1}^n \beta_i(u) u_x^i dx. \tag{69}$$

The coefficient of  $u_{xx}^i$  of the  $i$ th component of  $X$  is given by  $-\frac{3}{2} f^i c_i$ , while that of  $\tilde{X}$  equals zero. Thus we must have  $c_i = 0, i = 1, \dots, n$ . The theorem is proved.  $\square$

**Proof of Theorems 1 and 2.** Let us assume that the hydrodynamic bihamiltonian structure  $(\omega_1, \omega_2)$  has two  $N$ th order quasitrivial deformations of the form

$$\left( \omega_1, \omega_2 + \sum_{m=1}^N \epsilon^m P_m \right) + \mathcal{O}(\epsilon^{N+1}), \tag{70}$$

$$\left( \omega_1, \omega_2 + \sum_{m=1}^N \epsilon^m P_m + \epsilon^N Q \right) + \mathcal{O}(\epsilon^{N+1}). \tag{71}$$

Here  $P_m \in \hat{\Lambda}_{m+2}^2, Q \in \Omega_{N+2}^2$ . Due to our assumption, we can find a quasi-Miura transformation of the form (5) that transforms the bihamiltonian structure (70) to  $(\omega_1, \omega_2) + \mathcal{O}(\epsilon^{N+1})$ . Then this same quasi-Miura transformation transforms the bihamiltonian structure (71) to

$$(\omega_1, \omega_2 + \epsilon^N Q) + \mathcal{O}(\epsilon^{N+1}). \tag{72}$$

It is also a quasitrivial deformation of the bihamiltonian structure  $(\omega_1, \omega_2)$ , so we are able to find a quasi-Miura transformation that transforms  $(\omega_1, \omega_2)$  to (72). Such a quasi-Miura transformation can be represented by some vector fields  $Y_1, \dots, Y_N$  in the form

$$\begin{aligned} \omega_1 &= e^{-\epsilon^N \text{ad}_{Y_N}} \dots e^{-\epsilon \text{ad}_{Y_1}} \omega_1 + \mathcal{O}(\epsilon^{N+1}), \\ \omega_2 + \epsilon^N Q &= e^{-\epsilon^N \text{ad}_{Y_N}} \dots e^{-\epsilon \text{ad}_{Y_1}} \omega_2 + \mathcal{O}(\epsilon^{s+1}). \end{aligned} \tag{73}$$

From the above identities it follows that  $\partial_1 Y_N = 0, Q + \partial_2 Y_N = 0$ , so there exists a functional  $I$  such that  $Y_N = \partial_1 I, Q = \partial_1 \partial_2 I$ . On the other hand, the compatibility of  $(\omega_1, \omega_2 + \epsilon^N Q) + \mathcal{O}(\epsilon^{N+1})$  implies the existence of a vector field  $X \in \hat{\Lambda}_N^1$  satisfying  $Q = \partial_1 X$ . From the above two expressions of  $Q$  we see that we can express the vector field  $X$  as

$$X = \partial_2 I - \partial_1 J$$

with certain functional  $J \in \Omega$ .

Now the results of Theorem 3 lead to the following conclusions:

1. If  $N \neq 2$ , then  $I$  and  $J$  must be differential polynomials, so the two deformations (70) and (71) are related by a Miura transformation

$$u \mapsto u - \epsilon^N \partial_1 \partial_2 I. \tag{74}$$

Thus we proved **Theorem 1**.

2. Any second order deformation  $(\omega_1, \omega_2 + \epsilon P_1 + \epsilon^2 P_2) + \mathcal{O}(\epsilon^3)$  is equivalent to a second order deformation of the form  $(\omega_1, \omega_2 + \epsilon^2 \tilde{P}_2) + \mathcal{O}(\epsilon^3)$ . By applying the results of **Theorem 3** to the case with  $N = 2$ , we see that modulo a Miura transformation the deformed bihamiltonian structure can be represented in the form

$$(\omega_1, \omega_2 + \epsilon^2 \partial_1 (\partial_2 I - \partial_1 J)) + \mathcal{O}(\epsilon^3) \tag{75}$$

for some functionals  $I, J$  defined by

$$I = - \int \sum_{i=1}^n c_i(u^i) u_x^i \log u_x^i dx, \quad J = - \int u^i c_i(u^i) u_x^i \log u_x^i dx. \tag{76}$$

On the other hand, it is easy to see that any functionals  $I, J$  of the above form define a second order quasitrivial deformation of the bihamiltonian structure  $(\omega_1, \omega_2)$ . **Theorem 2** is proved.

From the proof of the main theorems it follows that any equivalence class of quasitrivial infinitesimal deformations of the bihamiltonian structure  $(\omega_1, \omega_2)$  has a unique representative of the forms (75) and (76) which corresponds to an element of the modified cohomology  $\hat{H}^2$ .

### 5. Some examples

In this section, we consider as examples the deformations of the bihamiltonian structures of hydrodynamic type that are related to the KdV and the nonlinear Schrödinger equations, these deformations yield the bihamiltonian structures for the Camassa–Holm hierarchy [2,3,18–20] and its generalization.

Let us first consider deformations of the bihamiltonian structure (3). The class of deformations that corresponds to the element of  $\hat{H}^2$  (see **Theorem 3**) with  $c(u) = \frac{1}{24}$  has a representative

$$\begin{aligned} \{u(x), u(y)\}_1 &= \delta'(x - y), \\ \{u(x), u(y)\}_2 &= u(x)\delta'(x - y) + \frac{1}{2}u(x)'\delta(x - y) + \frac{1}{8}\epsilon^2\delta'''(x - y). \end{aligned} \tag{77}$$

Here we redenote  $u^1 = u, c_1(u) = c(u)$ . It is just the well-known bihamiltonian structure for the KdV hierarchy [21,27,29]. Now if we take  $c(u) = \frac{1}{24}u$ , then the corresponding class of deformations has the following representative:

$$\begin{aligned} \{u(x), u(y)\}_1 &= \delta'(x - y) - \frac{\epsilon^2}{8}\delta'''(x - y), \\ \{u(x), u(y)\}_2 &= u(x)\delta'(x - y) + \frac{1}{2}u(x)'\delta(x - y). \end{aligned} \tag{78}$$

In fact, it is equivalent to the bihamiltonian structure

$$(\omega_1, \omega_2 + \epsilon^2 \partial_1(\partial_2 I - \partial_1 J)) + \mathcal{O}(\epsilon^3) \quad (79)$$

under the Miura transformation

$$u \mapsto u + \frac{\epsilon^2}{16} u''.$$

Here  $(\omega_1, \omega_2)$  denotes the bihamiltonian structure (3) and the functionals  $I$  and  $J$  are defined by

$$I = -\frac{1}{24} \int u(x) u'(x) \log u'(x) dx, \quad J = -\frac{1}{24} \int u(x)^2 u'(x) \log u'(x) dx.$$

The related bihamiltonian hierarchy of integrable systems is the Camassa–Holm hierarchy that is well known in soliton theory. It can be expressed by the following bihamiltonian recursion relations:

$$\frac{\partial u}{\partial t^q} = \{u(x), H_q\}_1 = \frac{2}{2q+1} \{u(x), H_{q-1}\}_2, \quad q \geq 0. \quad (80)$$

Here we start from the Casimir  $H_{-1} = \int u(x) dx$  of the first Poisson bracket, and then determine the Hamiltonians  $H_q$ ,  $q \geq 0$  recursively from the above relation. The recursive procedure of finding the Hamiltonians  $H_q$  is guaranteed by the triviality of the first Poisson cohomology of the Poisson structure  $\omega_1$  [7,14,22]. The first nontrivial flow  $\frac{\partial}{\partial t} = \frac{\partial}{\partial t^1}$  of the hierarchy can be put into the form

$$\left( v - \frac{\epsilon^2}{8} v_{xx} \right)_t = v v_x - \frac{\epsilon^2}{12} v_x v_{xx} - \frac{\epsilon^2}{24} v v_{xxx}. \quad (81)$$

Here the dependent variable  $v$  is defined by

$$u = v - \frac{1}{8} \epsilon^2 v_{xx}. \quad (82)$$

If we change the time variable as  $t^1 \mapsto t = -\frac{1}{3} t^1$  and put  $\epsilon^2 = 8$ , then the resulting equation is just the Camassa–Holm shallow water wave equation [2,3,18–20], which possesses most of the important properties of an integrable system. In particular, it has the following Lax pair representation

$$\epsilon^2 \phi_{xx} = \left( 2 - \frac{8v - \epsilon^2 v_{xx}}{2\lambda} \right) \phi, \quad (83)$$

$$\phi_t = \frac{1}{3} (\lambda + v) \phi_x - \frac{v_x}{6} \phi \quad (84)$$

and its initial value problems can be solved by using the inverse scattering method. The Camassa–Holm equation also possesses some features that are distinguished from the usual KdV-type integrable systems, such as the existence of peaked solitons, the nonlinear dependence of the arguments of its algebraic-geometric solutions on the spatial variable  $x$  [1]

and the nonexistence of tau function [14]. We will call Eq. (81) and the hierarchy (80) the Camassa–Holm equation and the Camassa–Holm hierarchy respectively.

The quasitriviality of the bihamiltonian structure (77) is proved in [14] and that of (78) can also be deduced by using the approach of [14].

For the choice of a general smooth function  $c(u)$ , we do not have at this moment an explicit expression of the correspondent class of deformations of the bihamiltonian structure (3). At the approximation up to  $\epsilon^4$  Lorenzoni obtained the expression of a representative of the corresponding class of deformations, and we can in fact go further to show that his result can be modified to reach the approximation up to higher orders of  $\epsilon$ . This fact strongly indicates the existence of a full deformation of the bihamiltonian structure (3) for any smooth function  $c(u)$ .

We now consider the deformations of the following bihamiltonian structure:

$$\{w_1(x), w_1(y)\}_1 = \{w_2(x), w_2(y)\}_1 = 0, \quad \{w_1(x), w_2(y)\}_1 = \delta'(x - y), \quad (85)$$

$$\{w_1(x), w_1(y)\}_2 = 2\delta'(x - y),$$

$$\{w_1(x), w_2(y)\}_2 = w_1(x)\delta'(x - y) + w'_1(x)\delta(x - y),$$

$$\{w_2(x), w_2(y)\}_2 = [w_2(x)\partial_x + \partial_x w_2(x)]\delta(x - y). \quad (86)$$

It is related to the Frobenius manifold with potential [5]

$$F = \frac{1}{2}w_1^2 w_2 + \frac{1}{2}w_2^2(\log w_2 - \frac{3}{2}).$$

The canonical coordinates of this bihamiltonian structure are given by

$$u^{1,2} = w_1 \pm 2\sqrt{w_2}. \quad (87)$$

Let us consider the following two classes of deformations:

*Case 1.* We take the element of  $\hat{H}^2$  with  $c_1(u) = c_2(u) = \frac{1}{24}$ , then the corresponding class of deformations has a representative

$$\{w_1(x), w_1(y)\}_1 = \{w_2(x), w_2(y)\}_1 = 0, \quad \{w_1(x), w_2(y)\}_1 = \delta'(x - y), \quad (88)$$

$$\{w_1(x), w_1(y)\}_2 = 2\delta'(x - y),$$

$$\{w_1(x), w_2(y)\}_2 = w_1(x)\delta'(x - y) + w'_1(x)\delta(x - y) - \epsilon\delta''(X - Y),$$

$$\{w_2(x), w_2(y)\}_2 = [w_2(x)\partial_x + \partial_x w_2(x)]\delta(x - y). \quad (89)$$

To see this, let us denote by  $\omega_1, \omega_2$  the two bivectors of the bihamiltonian structures (85) and (86), and by  $I, J$  the functionals

$$I = - \int \frac{1}{24}(u_x^1 \log u_x^1 + u_x^2 \log u_x^2) dx,$$

$$J = - \int \frac{1}{24}(u^1 u_x^1 \log u_x^1 + u^2 u_x^2 \log u_x^2) dx, \quad (90)$$

then by a direct computation it can be verified that the bihamiltonian structures (88) and (89) is equivalent to the bihamiltonian structure

$$(\omega_1, \omega_2 + \epsilon^2 \partial_1(\partial_2 I - \partial_1 J)) + \mathcal{O}(\epsilon^3) \tag{91}$$

under the Miura transformation

$$\begin{aligned} w_1 &\mapsto w_1 + \frac{\epsilon}{2\sqrt{3}} \frac{w_{2,x}}{w_2} + \epsilon^2 \left( \frac{1}{12} - \frac{1}{4\sqrt{3}} \right) \left( \frac{w_{1,xx}}{w_2} - \frac{w_{1,x}w_{2,x}}{w_2^2} \right), \\ w_2 &\mapsto w_2 + \epsilon \left( -\frac{1}{2} + \frac{1}{2\sqrt{3}} \right) w_{1,x}. \end{aligned} \tag{92}$$

The bihamiltonian hierarchy of integrable systems that is related to this bihamiltonian structure is called the *extended NLS hierarchy*, the algebraic properties of this hierarchy together with its relation to the  $CP^1$  topological sigma model is studied in detail in [5,15]. It is also shown in [5] that this hierarchy is equivalent to the extended Toda hierarchy [23,30] which contains the standard Toda lattice hierarchy.

Case 2. Let us take the element of  $\hat{H}^2$  with  $c_1(u) = \frac{(u^1)^2}{24}$ ,  $c_2(u) = \frac{(u^2)^2}{24}$ , then the correspondent class of deformations has a representative of the form

$$\begin{aligned} \{w_1(x), w_1(y)\}_1 &= \{w_2(x), w_2(y)\}_1 = 0, \\ \{w_1(x), w_2(y)\}_1 &= \delta'(x - y) - \epsilon \delta''(x - y), \end{aligned} \tag{93}$$

$$\begin{aligned} \{w_1(x), w_1(y)\}_2 &= 2\delta'(x - y), \\ \{w_1(x), w_2(y)\}_2 &= w_1(x)\delta'(x - y) + w'_1(x)\delta(x - y), \\ \{w_2(x), w_2(y)\}_2 &= [w_2(x)\partial_x + \partial_x w_2(x)]\delta(x - y). \end{aligned} \tag{94}$$

Denote by  $I, J$  the functionals

$$\begin{aligned} I &= - \int \frac{1}{24} ((u^1)^2 u_x^1 \log u_x^1 + (u^2)^2 u_x^2 \log u_x^2) dx, \\ J &= - \int \frac{1}{24} ((u^1)^3 u_x^1 \log u_x^1 + (u^2)^3 u_x^2 \log u_x^2) dx, \end{aligned} \tag{95}$$

then it can be verified that the bihamiltonian structures (93) and (94) is equivalent to the bihamiltonian structure:

$$(\omega_1, \omega_2 + \epsilon^2 \partial_1(\partial_2 I - \partial_1 J)) + \mathcal{O}(\epsilon^3)$$

modulo a Miura transformation of the form

$$\begin{aligned} w_1 &\mapsto w_1 + \epsilon^2 \left( \frac{w_1^2 + 4w_2}{24w_2} w_{1,x} \right)_x + \mathcal{O}(\epsilon^3), \\ w_2 &\mapsto w_2 + \epsilon \left( \frac{w_1^2}{4} - w_2 \right)_x - \epsilon^2 \left( \left( \frac{w_1^2 + 4w_2}{24w_2} - 1 \right) w_{2,x} \right)_x + \mathcal{O}(\epsilon^3). \end{aligned}$$

A hierarchy of integrable systems can be obtained by using the bihamiltonian recursion relation

$$\{w_i(x), H_{q-1}\}_2 = (q + 1)\{w_i(x), H_q\}_1, \quad q \geq 0. \tag{96}$$

Here we start from the Casimir  $H_{-1} = \int w_2(x) dx$  of the first Poisson bracket, and then determine the Hamiltonians  $H_q, q \geq 0$  recursively by using the above relation. The flows of the bihamiltonian hierarchy is then given by

$$\frac{\partial w_i}{\partial t^q} = \{w_i(x), H_q\}_1, \quad q \geq 0. \tag{97}$$

The first flow  $\frac{\partial}{\partial t^0}$  corresponds to the translation along the spatial variable  $x$ , and the second flow  $\frac{\partial}{\partial t} = \frac{\partial}{\partial t^1}$  has the form

$$(\varphi_1 - \epsilon\varphi_{1,x})_t = (\varphi_2 + \frac{1}{2}\varphi_1^2 - \frac{1}{2}\epsilon\varphi_1\varphi_{1,x})_x, \tag{98}$$

$$(\varphi_2 + \epsilon\varphi_{2,x})_t = (\varphi_1\varphi_2 + \frac{1}{2}\epsilon\varphi_1\varphi_{2,x})_x. \tag{99}$$

Here  $\varphi_1, \varphi_2$  are defined by  $w_1 = \varphi_1 - \epsilon\varphi_{1,x}, w_2 = \varphi_2 + \epsilon\varphi_{2,x}$ . By introducing the new variables

$$v_1 = \varphi_1, \quad v_2 = \varphi_2 + \epsilon\varphi_{2,x} - \frac{1}{4}(\varphi_1 - \epsilon\varphi_{1,x})^2$$

we can rewrite the above system of equations in the following form:

$$(v_1 - \epsilon^2 v_{1,xx})_t = (v_2 + \frac{3}{4}v_1^2 - \epsilon^2(\frac{1}{2}v_1 v_{1,xx} + \frac{1}{4}v_{1,x}^2))_x, \tag{100}$$

$$v_{2,t} = \frac{1}{2}v_1 v_{2,x} + v_2 v_{1,x}. \tag{101}$$

It easily follows from the above expression that the system of equations (100) and (101) is reduced to the Camassa–Holm equation (81) under the constraint

$$v_2 = 0 \tag{102}$$

together with the rescaling  $t \mapsto \frac{3}{2}t, \epsilon^2 \mapsto \frac{1}{8}\epsilon^2$ . So we can view the hierarchy (97) as a natural two-component generalization of the Camassa–Holm hierarchy (80). The following Lax pair formalism of the systems (100) and (101) manifests the above observation:

$$\epsilon^2 \phi_{xx} = \left( \frac{1}{4} - \frac{v_1 - \epsilon^2 v_{1,xx}}{2\lambda} - \frac{v_2}{\lambda^2} \right) \phi, \tag{103}$$

$$\phi_t = \frac{1}{2}(\lambda + v_1)\phi_x - \frac{1}{4}v_{1,x}\phi. \tag{104}$$

When we put  $v_2 = 0$  this Lax pair is reduced to the one that is given in (83) and (84).

The quasitriviality of the bihamiltonian structures (88) and (89) can be verified by using the method given in [14]. However, at this moment we do not have a proof for the quasitriviality of the bihamiltonian structures (93) and (94). In order to use the approach of [14] to prove its quasitriviality we need to construct a bihamiltonian



hierarchy of the form (97) that corresponds to the Casimir  $\int w_1(x) dx$  of the first Poisson bracket, since this functional is also a Casimir of the second Poisson bracket, the usual bihamiltonian recursion procedure fails to yield the needed Hamiltonians in a direct way.

## 6. Concluding remarks

For Any semisimple bihamiltonian structure of hydrodynamic type, we classify its infinitesimal quasitrivial deformations. We show that the equivalence classes of its second order quasitrivial deformations are parameterized by  $n$  arbitrary functions of one variable, and we prove that any class of its quasitrivial deformations is uniquely determined by its corresponding class of second order deformations. We end this paper with the following two remarks.

**Remark 1.** At a first glance the condition of quasitriviality seems to be highly nontrivial, however, a careful study shows that any deformation of the semisimple bihamiltonian structure of the forms (42) and (43) is quasitrivial at least for the case of  $n = 1$ , this fact together with the quasitriviality of any tau-symmetric bihamiltonian structure [14] indicates the validity of quasitriviality for any deformation of the semisimple bihamiltonian structure of the forms (42) and (43). An even more optimistic conjecture is the existence of a full deformation of a semisimple bihamiltonian structure of hydrodynamic type with a given second order deformation.

**Remark 2.** On the formal loop space of any semisimple Frobenius manifold there is defined a semisimple bihamiltonian structure of hydrodynamic type [9], a class of deformations of such bihamiltonian structure was constructed in [14], these deformations correspond to the element of the second cohomology  $\hat{H}^2$  with  $c_1 = \dots = c_n = \frac{1}{24}$ , they are compatible with the universal identities satisfied by the Gromov–Witten invariants of smooth projective varieties, for this reason we call them the topological deformations. The corresponding bihamiltonian hierarchy of integrable systems satisfies, in the sense of [14], the properties of tau-symmetry and linearization of the Virasoro symmetries. If we drop the requirement of linearization of the Virasoro symmetries, then the resulting tau symmetric bihamiltonian structure must correspond to an element of the second cohomology  $\hat{H}^2$  with constant  $c_1(u) = c_1, \dots, c_n(u) = c_n$ . An example of such bihamiltonian structures is given by the one that is obtained by using the Drinfeld–Sokolov construction for the affine Lie algebra of type  $B_2$  [4,13,16], in this case the corresponding element of the second cohomology  $\hat{H}^2$  is determined by the constant functions  $c_1 = \frac{1}{6}, c_2 = \frac{1}{12}$ .

## Acknowledgments

The authors are grateful to Boris Dubrovin for helpful suggestions and comments on the research of the subject. The researches of YZ were partially supported by the Chinese National Science Fund for Distinguished Young Scholars grant No.10025101 and the Special Funds of Chinese Major Basic Research Project “Nonlinear Sciences”.

## References

- [1] M.S. Alber, R. Camassa, Yu.N. Fedorov, D.D. Holm, J.E. Marsden, The complex geometry of weak piecewise smooth solutions of integrable nonlinear PDEs of shallow water and Dym type, *Commun. Math. Phys.* 221 (2001) 197–227.
- [2] R. Camassa, D.D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* 71 (1993) 1661–1664.
- [3] R. Camassa, D.D. Holm, J.M. Hyman, A new integrable shallow water equation, *Adv. Appl. Mech.* 31 (1994) 1–33.
- [4] P. Casati, M. Pedroni, Drinfeld–Sokolov reduction on a simple Lie algebra from the bihamiltonian point of view, *Lett. Math. Phys.* 25 (1992) 89–101.
- [5] G. Carlet, B. Dubrovin, Y. Zhang, The extended Toda hierarchy, *nlin-SI/0306060*, *Moscow Math. J.* 4 (2) (2004) 313.
- [6] P. Dedecker, W.M. Tulczyjev, Spectral sequences and the inverse problem of the calculus of variations, *Lecture Notes Math.* 836 (1980) 498–503.
- [7] L. Degiovanni, F. Magri, V. Sciacca, On deformation of Poisson manifolds of hydrodynamic type. *nlin.SI/010352*.
- [8] L.A. Dickey, *Soliton Equations and Hamiltonian Systems*, World Scientific, Singapore, 1991, p. 71.
- [9] B. Dubrovin, Geometry of 2D topological field theories, in: M. Francaviglia, S. Greco (Eds.), *Integrable Systems and Quantum Groups*, Montecatini, Terme, 1993. *Springer Lecture Notes in Mathematics* 1620 (1996) 120–348.
- [10] B. Dubrovin, S.P. Novikov, The Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov–Whitham averaging method, *Sov. Math. Dokl.* 270 (4) (1983) 665–669.
- [11] B. Dubrovin, S.P. Novikov, On Poisson brackets of hydrodynamic type, *Sov. Math. Dokl.* 279 (2) (1984) 294–297.
- [12] B. Dubrovin, S.P. Novikov, Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory, *Russ. Math. Surv.* 44 (1989) 35–124.
- [13] B. Dubrovin, Y. Zhang, Bihamiltonian hierarchies in 2D topological field theory at one-loop approximation, *Commun. Math. Phys.* 198 (1998) 311–361.
- [14] B. Dubrovin, Y. Zhang, Normal forms of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants. *math.DG/0108160*.
- [15] B. Dubrovin, Y. Zhang, Virasoro symmetries of the extended Toda hierarchy, *math.DG/0308152*, *Commun. Math. Phys.* 250 (2004) 161–193.
- [16] T. Eguchi, Y. Yamada, S.-K. Yang, On the genus expansion in the topological string theory, *Rev. Math. Phys.* 7 (1995) 279–309.
- [17] E.V. Ferapontov, Compatible Poisson brackets of hydrodynamic type, *J. Phys. A* 34 (11) (2001) 2377–2388.
- [18] A.S. Fokas, On a class of physically important integrable equations, *Physica D* 87 (1995) 145–150.
- [19] B. Fuchssteiner, Some tricks from the symmetry-toolbox for nonlinear equations: generalizations of the Camassa–Holm equation, *Physica D* 95 (1996) 229–243.
- [20] B. Fuchssteiner, A.S. Fokas, Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Physica D* 4 (1981) 47–66.
- [21] C.S. Gardner, Korteweg–de Vries equation and generalizations IV, *J. Math. Phys.* 12 (1971) 1548–1551.
- [22] E. Getzler, A Darboux theorem for Hamiltonian operators in the formal calculus of variations, *Duke Math. J.* 111 (2002) 535–560.
- [23] E. Getzler, *The Toda conjecture, Symplectic Geometry and Mirror Symmetry* (Seoul, 2000), World Scientific Publishing, River Edge, NJ, 2001, pp. 51–79. *math.AG/0108108*.
- [24] A. Lichnerowicz, Les varietes de Poisson et leurs algèbres de Lie associées, *J. Diff. Geom.* 12 (1977) 253–300.
- [25] P. Lorenzoni, Deformations of bihamiltonian structures of hydrodynamic type, *J. Geom. Phys.* 44 (2002) 331–375.
- [26] A. Maltsev, The conservation of the Hamiltonian structures in Whitham’s method of averaging, *Izvestiya, Mathematics* 63 (6) (1999) 1171–1201.

- [27] F. Magri, A simple construction of integrable systems, *J. Math. Phys.* 19 (1978) 1156–1162.
- [28] O.I. Mokhov, Compatible and almost compatible pseudo-Riemannian metrics, *Funct. Anal. Appl.* 35 (2) (2001) 100–110.
- [29] V.E. Zakharov, L.D. Faddeev, Korteweg–de Vries equation is a completely integrable Hamiltonian system, *Funkz. Anal. Priloz.* 5 (1971) 18–27.
- [30] Y. Zhang, On the  $CP^1$  topological sigma model and the Toda lattice hierarchy, *J. Geom. Phys.* 40 (2002) 215–232.